Difference method for some finite algebraic series

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(A) If the general term, u_r, of the series is in "product" form, you can **add** one more factor to the end of the general term u_r, so as to form a sequence v_r and then apply the difference method.

Example 1

Find the sum of n terms of the series : $1 \cdot 4 \cdot 7 + 4 \cdot 7 \cdot 10 + 7 \cdot 10 \cdot 13 + ...$

Solution

(1) Find the general term of the given series:

$$u_r = (3r-2)(3r+1)(3r+4) \qquad , r \ge 1 .$$

Note : The general term is **not** $u_r = r(r+3)(r+6)$

(2) Form another sequence v_r by adding one more factor to the end of the general term u_r :

$$v_r = (3r-2)(3r+1)(3r+4)(3r+7) \qquad , r \ge 1 .$$

(3) Find v_{r-1} :

$$v_{r-1} = (3r-5)(3r-2)(3r+1)(3r+4) \qquad , r \ge 2 .$$

(4) Form the difference :

$$v_r - v_{r-1} = (3r-2)(3r+1)(3r+4)(3r+7) - (3r-5)(3r-2)(3r+1)(3r+4)$$

$$= (3r-2)(3r+1)(3r+4)[(3r+7)-(3r-5)]$$

$$= 12(3r-2)(3r+1)(3r+4) = 12u_r \qquad , r \ge 2 .$$

(5) Find the sum :

$$12\sum_{r=1}^{n} u_r = 12\sum_{r=2}^{n} u_r + 12u_1 = \sum_{r=2}^{n} (v_r - v_{r-1}) + 12u_1 \quad , \quad \text{note that the difference is good for } r \ge 2 .$$

$$= [(v_n - v_{n-1}) + (v_{n-1} - v_{n-2}) + ...(v_3 - v_2) + (v_2 - v_1)] + 12u_1$$

$$= (v_n - v_1) + 12u_1$$

$$= (3n-2)(3n+1)(3n+4)(3n+7) - 1 \cdot 4 \cdot 7 \cdot 10 + 12(1 \cdot 4 \cdot 7)$$

$$\therefore \quad \sum_{r=1}^{n} u_r = \frac{1}{12}[(3n-2)(3n+1)(3n+4)(3n+7) + 56]$$

(6) It is more convenient to define v_0. Since $v_1 - v_0 = 12\, u_1$, we have:

$1 \cdot 4 \cdot 7 \cdot 10 - v_0 = 12 \cdot 1 \cdot 4 \cdot 7$ and we define $v_0 = -56$ (or $(-2) \cdot 1 \cdot 4 \cdot 7$) .

$$12\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} (v_r - v_{r-1}) = v_n - v_0 = (3n-2)(3n+1)(3n+4)(3n+7) + 56$$

$$\therefore \quad \sum_{r=1}^{n} u_r = \frac{1}{12}[(3n-2)(3n+1)(3n+4)(3n+7) + 56]$$

Exercise : Show that : $1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + ...(\text{to } n \text{ terms}) = n(n+2)(2n^2 + 4n - 1)$

(B) If the general term, u_r, is in "quotient" form, you can **remove** one more factor at the end of the general term u_r, so as to form a sequence v_r and then apply the difference method.

Example 2

Find the sum of n terms of the series : $\frac{1}{1\cdot 4\cdot 7}+\frac{1}{4\cdot 7\cdot 10}+\frac{1}{7\cdot 10\cdot 13}+\ldots$

Solution

(1) Find the general term of the given series:

$$u_r=\frac{1}{(3r-2)(3r+1)(3r+4)} \qquad , r\ge 1 .$$

(2) Form another sequence v_r by removing one factor at the end of the general term u_r :

$$v_r=\frac{1}{(3r+1)(3r+4)} \qquad , r\ge 1 .$$

(3) Find v_{r-1} : $\quad v_{r-1}=\frac{1}{(3r-2)(3r+1)} \qquad , r\ge 1$

Note that $v_0=\frac{1}{4}$ is well-defined and that is why in (2), we remove the first factor in the denominator. Removing $(3r + 4)$ instead in the denominator will get v_0 undefined .

(4) Form the difference :

$$\begin{aligned} v_r-v_{r-1} &= \frac{1}{(3r+1)(3r+4)}-\frac{1}{(3r-2)(3r+1)} \\ &= \frac{1}{(3r-2)(3r+1)(3r+4)}[(3r-2)-(3r+4)] \\ &= -6u_r \qquad , r\ge 2 . \end{aligned}$$

(5) Find the sum :

$$-6\sum_{r=1}^{n} u_r=\sum_{r=1}^{n}(v_r-v_{r-1})=v_n-v_0=\frac{1}{(3n+1)(3n+4)}-\frac{1}{4}$$

$$\therefore \quad \sum_{r=1}^{n} u_r=\frac{1}{6}\left[\frac{1}{4}-\frac{1}{(3n+1)(3n+4)}\right]$$

Exercise :

Prove that :

(1) $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$

(2) $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)(2r+3)}=\frac{1}{4}\left[\frac{1}{3}-\frac{1}{(2n+1)(2n+3)}\right]$

(C) More complicate finite series can be broken into two (or more) series and each can be handled by the methods above .

Example 3

Find the sum of n terms of the series : $1\cdot4\cdot7\cdot8+4\cdot7\cdot10\cdot11+7\cdot10\cdot13\cdot14+\ldots$

Hint

The general term,

$$\begin{aligned} u_r &= (3r-2)(3r+1)(3r+4)(3r+5) \\ &= (3r-2)(3r+1)(3r+4)[(3r+7)-2] \\ &= (3r-2)(3r+1)(3r+4)(3r+7)-2(3r-2)(3r+1)(3r+4) \end{aligned}$$

Answer $\frac{1}{10}(3n-2)(3n+1)(3n+4)(3n+7)(2n+5)+28$

Example 4

Find the sum of n terms of the series : $\frac{3}{2\cdot3\cdot4\cdot5}+\frac{5}{3\cdot4\cdot5\cdot6}+\frac{7}{4\cdot5\cdot6\cdot7}+\ldots$

Hint

The general term,

$$\begin{aligned} u_r &= \frac{2r+1}{(r+1)(r+2)(r+3)(r+4)} \\ &= \frac{2(r+1)-1}{(r+1)(r+2)(r+3)(r+4)} = \frac{2}{(r+2)(r+3)(r+4)} - \frac{1}{(r+3)(r+4)} \end{aligned}$$

Answer $\frac{5}{72}-\frac{1}{3}\frac{3n+5}{(n+2)(n+3)(n+4)}$

(D) If the general term, u_r , of the series is in "product + quotient" form with same number of factors in numerator and denominator, you can **add** one more factor to the end of the general term u_r , so as to form a sequence v_r and then apply the difference method.

Example 5

Find the sum of n terms of the series : $\frac{2}{1}+\frac{2\cdot4}{1\cdot3}+\frac{2\cdot4\cdot6}{1\cdot3\cdot5}+\ldots$

Hint $u_r = \frac{2\cdot4\cdot\ldots\cdot(2r)}{1\cdot3\cdot\ldots\cdot(2r-1)}$, $v_r = \frac{2\cdot4\cdot\ldots\cdot(2r)(2r+2)}{1\cdot3\cdot\ldots\cdot(2r-1)}$

Answer $\frac{1}{3}\left\{\frac{2\cdot4\cdot\ldots\cdot(2r)(2n+2)}{1\cdot3\cdot\ldots\cdot(2n-1)}-2\right\}$